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Free field realization of vertex operators for level two modules of $U_q(\hat{\mathfrak{sl}}(2))$

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Abstract. Free field realization of vertex operators for level-two modules of $U_q(\hat{\mathfrak{sl}}(2))$ is shown through the free field realization of the modules given by M Idzumi (*Int. J. Mod. Phys.* A **9** 4449 *Preprint* hep-th/9307129). We constructed type I and II vertex operators when the spin of the associated evaluation module is $\frac{1}{2}$ and type IIs for the spin 1.

1. Introduction

Vertex operators for the quantum affine algebra $U_q(\hat{\mathfrak{sl}}(2))$ have played essential roles in the algebraic analysis of solvable lattice models since the pioneering works of [1–3]. In these works, which analyse the XXZ model, type I vertex operators are identified with half-infinite transfer matrices as their representation-theoretical counterpart and type II vertex operators are interpreted as particle creation operators. To perform concrete computation such as a trace of composition of vertex operators, we need free field realization of modules and operators. In the said example of the XXZ model, the integral expressions of n-point correlation functions which are special cases of the traces are obtained through bosonization of the level-one module of $U_q(\hat{\mathfrak{sl}}(2))$.

Motivated by these results, Idzumi [4, 5] constructed level-two modules and type I vertex operators accompanied by spin 1 evaluation modules for $U_q(\hat{\mathfrak{sl}}(2))$ in terms of bosons and fermions and then calculated correlation functions of a spin 1 analogue of the XXZ model. The purpose of this paper is to extend Idzumi's free field realization to other kinds of vertex operators i.e. type I and II vertex operators for the level-two modules associated with the evaluation module of spin $\frac{1}{2}$ and the type IIs for the spin 1. The results are given in section 3 and their derivation is discussed in the first case in section 4. The results together with [4, 5] give the complete set of vertex operators for the level-two module of $U_q(\hat{\mathfrak{sl}}(2))$ and enable one to calculate form factors of the spin 1 analogue of the XXZ model.

Recently Jimbo and Shiraishi [7] showed a coset-type construction for the deformed Virasoro algebra with the vertex operators for $U_q(\hat{\mathfrak{sl}}(2))$. They constructed a primary operator for the deformed Virasoro algebra as a coset-type composition of vertex operators which may be denoted as $(U_q(\hat{\mathfrak{sl}}(2))_k \oplus U_q(\hat{\mathfrak{sl}}(2))_1)/U_q(\hat{\mathfrak{sl}}(2))_{k+1}$. We hope that our results will be helpful for extending this work to the deformed supersymmetric Virasoro algebra through $(U_q(\hat{\mathfrak{sl}}(2))_k \oplus U_q(\hat{\mathfrak{sl}}(2))_2)/U_q(\hat{\mathfrak{sl}}(2))_{k+2}$.

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2. Free field realization of the level-two module

2.1. Convention

In the following we will use U to denote the quantum affine algebra $U_q(\hat{\mathfrak{sl}}(2))$. Unless otherwise mentioned, we follow the notations of [4, 5]. As for the free field representation, we slightly modify the convention.

The quantum affine algebra U is an associative algebra with unit 1 generated by $e_i, f_i (i = 0, 1), q^h (h \in P^*)$ with relations

$$q^0 = 1 \qquad q^h q^{h'} = q^{h+h'}$$
$$q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i \qquad q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i$$
$$[e_i, f_i]^\dagger = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}} \qquad (t_i = q^{h_i})$$
$$e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 = 0$$
$$f_i^3 f_j - [3] f_i^2 f_j f_i + [3] f_i f_j f_i^2 - f_j f_i^3 = 0$$

where $P = \mathbb{Z}\Lambda_0 + \mathbb{Z}\Lambda_1 + \mathbb{Z}\delta$ is the weight lattice of the affine Lie algebra $\hat{\mathfrak{sl}}(2)$ and P^* is the dual lattice to P with the dual basis $\{h_0, h_1, d\}$ to $\{\Lambda_0, \Lambda_1, \delta\}$ with respect to the natural pairing $\langle\, ,\, \rangle : P \times P^* \to \mathbb{Z}$. We also use current-type generators introduced by Drinfeld [11]

$$[a_k \cdot a_l] = \delta_{k+l,0} \frac{[2k]}{k} \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}$$
$$K a_k K^{-1} = a_k \qquad K x_k^{\pm} K^{-1} = q^{\pm 2} x_k^{\pm}$$
$$[a_k, x_l^{\pm}] = \pm \frac{[2k]}{k} \gamma^{\mp|k|/2} x_{k+l}^{\pm}$$
$$x_{k+l}^{\pm} x_l^{\pm} - q^{\pm 2} x_l^{\pm} x_{k+l}^{\pm} = q^{\pm 2} x_k^{\pm} x_{l+1}^{\pm} - x_{l+1}^{\pm} x_k^{\pm}$$
$$[x_k^+, x_l^-] = \frac{\gamma^{\frac{k-l}{2}} \psi_{k+l} - \gamma^{\frac{l-k}{2}} \phi_{k+l}}{q - q^{-1}}$$

where ψ_k, and φ_k are defined as

$$\sum_{k\geqslant 0} \psi_k z^{-k} = K \exp\left\{(q - q^{-1}) \sum_{k\geqslant 1} a_k z^{-k}\right\}$$
$$\sum_{k\geqslant 0} \phi_k z^{k} = K^{-1} \exp\left\{-(q - q^{-1}) \sum_{k\geqslant 1} a_{-k} z^{k}\right\}.$$

The relations between the two types of generators are

$$t_1 = K \qquad t_0 = \gamma K^{-1} \qquad e_1 = x_0^+, e_0 t_1 = x_1^- \qquad f_1 = x_0^- \qquad t_1^{-1} f_1 = x_0^{-1}.$$

The highest weight module and the evaluation module are described compactly in [4].

Commutation and anticommutation relations of bosons and fermions are given by

$$[a_m, a_n] = \delta_{m+n,0} \frac{[2m]^2}{m}$$
$$\{\phi_m, \phi_n\}^{\dagger\dagger} = \delta_{m+n,0} \eta_m$$
$$\eta_m = q^{2m} + q^{-2m}.$$

† [A, B] = AB − BA.
†† {A, B} = AB + BA.

where $m, n \in \mathbb{Z} + \frac{1}{2}$ or $\in \mathbb{Z}$ for the Neveu–Schwarz sector or Ramond sector, respectively. Fock spaces and vacuum vectors are denoted as $\mathcal{F}^a$, $\mathcal{F}^{\phi^{NS}}$, $\mathcal{F}^{\phi^R}$ and $|\mathrm{vac}\rangle$, $|NS\rangle$, $|R\rangle$ for the boson and NS and R fermion, respectively. Fermion currents are defined as

$$\phi^{NS}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \phi_n^{NS} z^{-n} \qquad \phi^R(z) = \sum_{n\in\mathbb{Z}} \phi_n^R z^{-n}.$$

$Q = \mathbb{Z}\alpha$ is the root lattice of $\mathfrak{sl}_2$ and $F[Q]$ is the group algebra. We use ∂ as

$$[\partial, \alpha] = 2.$$

2.2. $V(2\Lambda_0)$, $V(2\Lambda_1)$

The highest weight module $V(2\Lambda_0)$ is identified with the Fock space

$$\mathcal{F}_+^{(0)} = \mathcal{F}^a \otimes \{(\mathcal{F}_{\mathrm{even}}^{\phi^{NS}} \otimes F[2Q]) \oplus (\mathcal{F}_{\mathrm{odd}}^{\phi^{NS}} \otimes \mathrm{e}^\alpha F[2Q])\} \tag{1}$$

subscripts even and odd represent the number of fermions. The highest weight vector is $|\mathrm{vac}\rangle \otimes |NS\rangle \otimes 1$. $V(2\Lambda_1)$ is

$$\mathcal{F}_-^{(0)} = \mathcal{F}^a \otimes \{(\mathcal{F}_{\mathrm{even}}^{\phi^{NS}} \otimes \mathrm{e}^\alpha F[2Q]) \oplus (\mathcal{F}_{\mathrm{odd}}^{\phi^{NS}} \otimes F[2Q])\} \tag{2}$$

with the highest weight vector being $|\mathrm{vac}\rangle \otimes |NS\rangle \otimes \mathrm{e}^\alpha$. Note that

$$\mathcal{F}^{(0)} = \mathcal{F}_-^{(0)} \oplus \mathcal{F}_+^{(0)}$$
$$\mathcal{F}^{(0)} = \mathcal{F}^a \otimes \mathcal{F}^{\phi^{NS}} \otimes F[Q].$$

The operators are realized in the following manner:

$$\gamma = q^2 \qquad K = q^\partial$$
$$x^\pm(z) = \sum_{m\in\mathbb{Z}} x_m^\pm z^{-m} = E_<^\pm(z) E_>^\pm(z) \phi^{NS}(z) \mathrm{e}^{\pm\alpha} z^{\frac{1}{2}\pm\frac{1}{2}\partial}$$
$$E_<^\pm(z) = \exp\left(\pm \sum_{m>0} \frac{a_{-m}}{[2m]} q^{\mp m} z^m\right) \qquad E_>^\pm(z) = \exp\left(\mp \sum_{m>0} \frac{a_m}{[2m]} q^{\mp m} z^{-m}\right)$$

and

$$d = -\frac{\partial^2}{8} + \frac{(\lambda, \lambda)}{4} - \sum_{m=1}^{\infty} m N_m^a - \sum_{k>0} k N_k^{\phi^{NS}} \tag{3}$$

$$N_m^a = \frac{m}{[2m]^2} a_{-m} a_m \qquad N_k^{\phi^{NS}} = \frac{1}{\eta_m} \phi_{-m}^{NS} \phi_m^{NS} \qquad (m > 0) \tag{4}$$

where the highest weight vector of the module should be substituted for λ of (3).

2.3. $V(\Lambda_0 + \Lambda_1)$

The module $V(\Lambda_0 + \Lambda_1)$ is identified with

$$\mathcal{F}^{(1)} = \mathcal{F}^a \otimes \mathcal{F}^{\phi^R} \otimes \mathrm{e}^{\frac{\alpha}{2}} F[Q] \tag{5}$$

where

$$\phi_0^R |R\rangle = |R\rangle.$$

The highest weight vector is identified with $|\mathrm{vac}\rangle \otimes |R\rangle \otimes \mathrm{e}^{\frac{\alpha}{2}}$.

Operators are constructed in the same way as before except that subscripts for fermion sector are R instead of NS.

3. Free field realizations of vertex operators

Let V, V' be level-two modules and $V_z^{(k)}$ be a spin $k/2$ evaluation module of U. Vertex operators we will consider are U-linear maps of the following kinds [8, 9]

$$\Phi_V^{V',k}(z) : V \longrightarrow V' \otimes V_z^{(k)} \tag{6}$$

$$\Psi_V^{k,V'}(z) : V \longrightarrow V_z^{(k)} \otimes V'. \tag{7}$$

Vertex operators of the form (6), (7) are called type I and II, respectively. Components of vertex operators are defined as

$$\Phi(z)_V^{V',k} = \sum_{n=0}^{k} \Phi_n(z) \otimes u_n \qquad \Psi(z)_V^{k,V'} = \sum_{n=0}^{k} u_n \otimes \Psi_n(z).$$

3.1. *Type I vertex operators for level two and spin $\frac{1}{2}$*

We show free field realization of type I vertex operators of the following kinds

$$\Phi_{2\Lambda_i}^{\Lambda_0+\Lambda_1,1}(z) : V(2\Lambda_i) \longrightarrow V(\Lambda_0+\Lambda_1) \otimes V_z^{(1)} \tag{8}$$

$$\Phi_{\Lambda_0+\Lambda_1}^{2\Lambda_i,1}(z) : V(\Lambda_0+\Lambda_1) \longrightarrow V(2\Lambda_i) \otimes V_z^{(1)} \tag{9}$$

where $i = 0$ or 1.

Under the free field realization of level-two modules reviewed in secton 2, the explicit forms of the components of the vertex operators in (8) are

$$\Phi_1(z) = B_{I,<}(z) B_{I,>}(z) \Omega_{NS}^{R}(z) \mathrm{e}^{\alpha/2} (-q^4 z)^{\partial/4} \tag{10}$$

$$\Phi_0(z) = \oint \frac{\mathrm{d}w}{2\pi \mathrm{i}} B_{I,<}(z) E_<^-(w) B_{I,>}(z) E_>^-(w) \Omega_{NS}^{R}(z) \phi^{NS}(w) \mathrm{e}^{-\alpha/2} (-q^4 z)^{\partial/4}$$
$$\times w^{-\partial/2} (-q^4 z w^3)^{-\frac{1}{2}} \frac{\left(\frac{w}{q^3 z}; q^4\right)_\infty}{\left(\frac{w}{qz}; q^4\right)_\infty} \left\{ \frac{w}{1 - q^{-3} w/z} + \frac{q^5 z}{1 - q^5 z/w} \right\} \tag{11}$$

$$B_{I,<}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{[n] a_{-n}}{[2n]^2} (q^5 z)^n \right) \tag{12}$$

$$B_{I,>}(z) = \exp\left(- \sum_{n=1}^{\infty} \frac{[n] a_n}{[2n]^2} (q^3 z)^{-n} \right). \tag{13}$$

The integrand of $\Phi_0(z)$ has poles only at $w = q^5 z, q^3 z$ except for $w = 0, \infty$ and the contour of integration encloses $w = 0, q^5 z$, details are discussed in section 4. For those of (9) we just replace $\Omega_{NS}^{R}(z)$ with $\Omega_{R}^{NS}(z)$ in (10), (11).

The fermionic part $\Omega(z)$'s are maps between different fermion sectors and satisfy

$$\phi^{NS}(w) \Omega(z)_R^{NS} = \left(\frac{-q^4 z}{w} \right)^{1/2} \frac{\left(\frac{w}{q^3 z}; q^4\right)_\infty \left(\frac{q^7 z}{w}; q^4\right)_\infty}{\left(\frac{w}{qz}; q^4\right)_\infty \left(\frac{q^5 z}{w}; q^4\right)_\infty} \Omega(z)_R^{NS} \phi^R(w) \tag{14}$$

and exactly the same equation except subscripts for fermion sectors are exchanged. This kind of mapping for fermions first appeared in high-energy physics theory as 'fermion emission vertex operator' [6, 10]. Their free field realizations are

$$\Omega_{NS}^{R}(z) = \langle NS | \mathrm{e}^{Y} | R \rangle \tag{15}$$

$$Y=-\sum_{m>n\geqslant 0}X_{m,n}\varphi^R_{-m}\varphi^R_{-n}z^{m+n}-\sum_{k>l\geqslant 0}X_{k+1/2,l+1/2}\varphi^{NS}_{k+1/2}\varphi^{NS}_{l+1/2}z^{-k-l-1}$$
$$+\sum_{\substack{m\geqslant 0\\ k\geqslant 0}}X_{m,-k-1/2}\varphi^R_{-m}\varphi^{NS}_{k+1/2}z^{m-k-1/2} \tag{16}$$

$$\Omega^{NS}_R(z)=\langle R|\mathrm{e}^{Y'}|NS\rangle \tag{17}$$

$$Y'=\sum_{k>l\geqslant 0}X_{k+1/2,l+1/2}\varphi^{NS}_{-k-1/2}\varphi^{NS}_{-l-1/2}z^{k+l+1}+\sum_{m>n\geqslant 0}X_{m,n}\varphi^R_m\varphi^R_n z^{-m-n}$$
$$-\sum_{\substack{k\geqslant 0\\ m\geqslant 0}}X_{-k-1/2,m}\varphi^{NS}_{-k-1/2}\varphi^R_m z^{k-m+1/2} \tag{18}$$

$$\varphi^R_0=\phi^R_0 \qquad \varphi^R_{-m}=\phi^R_{-m}\frac{\gamma_m q^{5m}}{\eta_m} \qquad \varphi^R_m=\phi^R_m\frac{\gamma_m q^{-3m}}{\eta_m} \qquad (m>0) \tag{19}$$

$$\varphi^{NS}_{k+1/2}=\phi^{NS}_{k+1/2}\frac{\gamma_k q^{-3k-2}}{\eta_{k+1/2}}(-(-1)^{1/2}) \qquad \varphi^{NS}_{-k-1/2}=\phi^{NS}_{-k-1/2}\frac{\gamma_k q^{5k+2}}{\eta_{k+1/2}}(-1)^{1/2} \qquad (k>0) \tag{20}$$

$$X_{k,l}=\frac{q^{4k}-q^{4l}}{1-q^{4(k+l)}}$$
$$\gamma_n=\frac{(q^2;q^4)_n}{(q^4;q^4)_n} \qquad \frac{(q^2z;q^4)_\infty}{(z;q^4)_\infty}=\sum_{n=0}^{\infty}\gamma_n z^n. \tag{21}$$

(15), (17) are to mean that a matrix element is given by

$$_R\langle\mathrm{out}|\Omega^R_{NS}(z)|\mathrm{in}\rangle_{NS}={}_R\langle\mathrm{out}|\otimes\langle NS|\mathrm{e}^Y|R\rangle\otimes|\mathrm{in}\rangle_{NS}$$
$$\text{for } |\mathrm{out}\rangle_R\in\mathcal{F}^{\phi^R},\ |\mathrm{in}\rangle_{NS}\in\mathcal{F}^{\phi^{NS}}.$$

We define the normalized vertex operators $\tilde{\Phi}(z)$'s as follows

$$\langle\Lambda_0+\Lambda_1|\tilde{\Phi}_1(z)|2\Lambda_0\rangle=1 \qquad \langle 2\Lambda_1|\tilde{\Phi}_1(z)|\Lambda_0+\Lambda_1\rangle=1$$
$$\langle\Lambda_0+\Lambda_1|\tilde{\Phi}_0(z)|2\Lambda_1\rangle=1 \qquad \langle 2\Lambda_0|\tilde{\Phi}_0(z)|\Lambda_0+\Lambda_1\rangle=1$$

and these are given by

$$\tilde{\Phi}^{\Lambda_0+\Lambda_1,1}_{2\Lambda_0}(z)=\Phi(z) \tag{22}$$
$$\tilde{\Phi}^{2\Lambda_1,1}_{\Lambda_0+\Lambda_1}(z)=(-q^4z)^{-1/4}\Phi(z) \tag{23}$$
$$\tilde{\Phi}^{2\Lambda_0,1}_{\Lambda_0+\Lambda_1}(z)=(-q^4z)^{1/4}\Phi(z) \tag{24}$$
$$\tilde{\Phi}^{\Lambda_0+\Lambda_1,1}_{2\Lambda_1}(z)=(-q^6z)^{-1/2}\Phi(z). \tag{25}$$

3.2. Type II vertex operators for level two and spin $\frac{1}{2}$

We consider type II vertex operators of the following kind

$$\Psi^{1,\Lambda_0+\Lambda_1}_{2\Lambda_i}(z):V(2\Lambda_i)\longrightarrow V^{(1)}_z\otimes V(\Lambda_0+\Lambda_1) \tag{26}$$
$$\Psi^{1,2\Lambda_i}_{\Lambda_0+\Lambda_1}(z):V(\Lambda_0+\Lambda_1)\longrightarrow V^{(1)}_z\otimes V(2\Lambda_i). \tag{27}$$

Explicit forms of the components are as follows.

$$\Psi_0(z)=B_{II,<}(z)B_{II,>}(z)\Omega(q^{-2}z)\mathrm{e}^{-\alpha/2}(-q^2z)^{-\partial/4} \tag{28}$$

$$\Psi_1(z)=\oint\frac{\mathrm{d}w}{2\pi\mathrm{i}}B_{II,<}(z)E^+_<(w)B_{II,>}(z)E^+_>(w)\Omega(q^{-2}z)\phi(w)\mathrm{e}^{\alpha/2}(-q^2z)^{-\partial/4}$$
$$\times w^{\partial/2}(-q^2zw^3)^{-\frac{1}{2}}\frac{\left(\frac{w}{qz};q^4\right)_\infty}{\left(\frac{qw}{z};q^4\right)_\infty}\left\{\frac{w}{1-q^{-3}w/z}+\frac{q^3z}{1-qz/w}\right\} \tag{29}$$
$$B_{II,<}(z)=\exp\left(-\sum_{n=1}^{\infty}\frac{[n]a_{-n}}{[2n]^2}(qz)^n\right) \tag{30}$$
$$B_{II,>}(z)=\exp\left(\sum_{n=1}^{\infty}\frac{[n]a_{n}}{[2n]^2}(q^3z)^{-n}\right). \tag{31}$$

The integrand of $\Psi_1(z)$ has poles only at $w=q^3z, qz$ except for $w=0,\infty$ and the contour of integration encloses $w=0,qz$. Subscripts for fermion sectors are abbreviated.

Normalized vertex operators are defined by the conditions

$$\langle\Lambda_0+\Lambda_1|\tilde{\Psi}_1(z)|2\Lambda_0\rangle=1 \qquad \langle 2\Lambda_1|\tilde{\Psi}_1(z)|\Lambda_0+\Lambda_1\rangle=1$$
$$\langle\Lambda_0+\Lambda_1|\tilde{\Psi}_0(z)|2\Lambda_1\rangle=1 \qquad \langle 2\Lambda_0|\tilde{\Psi}_0(z)|\Lambda_0+\Lambda_1\rangle=1$$

and these are given by

$$\tilde{\Psi}^{1,\Lambda_0+\Lambda_1}_{2\Lambda_0}(z)=(-q)^{-1}\Psi(z) \tag{32}$$
$$\tilde{\Psi}^{1,2\Lambda_1}_{\Lambda_0+\Lambda_1}(z)=-(-q^6z)^{-1/4}\Psi(z) \tag{33}$$
$$\tilde{\Psi}^{1,2\Lambda_0}_{\Lambda_0+\Lambda_1}(z)=(-q^2z)^{1/4}\Psi(z) \tag{34}$$
$$\tilde{\Psi}^{1,\Lambda_0+\Lambda_1}_{2\Lambda_1}(z)=(-q^2z)^{1/2}\Psi(z). \tag{35}$$

3.3. *Type II vertex operators for level two and spin 1*

When the spin of the evaluation module is 1, the type II vertex operators do not contain any fermion emission vertex operators:

$$\Psi^{2,2\Lambda_i}_{2\Lambda_i}(z):V(2\Lambda_i)\longrightarrow V^{(2)}_z\otimes V(2\Lambda_i) \tag{36}$$
$$\Psi^{2,\Lambda_0+\Lambda_1}_{\Lambda_0+\Lambda_1}(z):V(\Lambda_0+\Lambda_1)\longrightarrow V^{(2)}_z\otimes V(\Lambda_0+\Lambda_1). \tag{37}$$

Explicit form of the components is as follows

$$\Psi_0(z)=F_{II,<}(z)F_{II,>}(z)\mathrm{e}^{-\alpha}(-q^2z)^{-\partial/2+1} \tag{38}$$
$$\Psi_1(z)=\oint\frac{\mathrm{d}w}{2\pi\mathrm{i}}F_{II,<}(z)E^+_<(w)F_{II,>}(z)E^+_>(w)\phi(w)$$
$$\times\left(\frac{w}{-q^2z}\right)^{\partial/2}w^{-1/2}\left\{\frac{1}{1-\frac{w}{q^4z}}+\frac{q^4z}{w\left(1-\frac{z}{w}\right)}\right\}. \tag{39}$$

The integration contour encircles poles $w=0,z$ but the pole $w=q^4z$ lies outside of it.

$$\Psi_2(z)=\oint\frac{\mathrm{d}w_2}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}w_1}{2\pi\mathrm{i}}F_{II,<}(z)E^+_<(w_1)E^+_<(w_2)F_{II,>}(z)E^+_>(w_1)E^+_>(w_2)$$
$$\times\mathrm{e}^{\alpha}\left(\frac{w_1w_2}{-q^2z}\right)^{\partial/2}(w_1w_2)^{-1/2}\left\{\frac{1}{1-\frac{w_1}{q^4z}}+\frac{q^4z}{w_1\left(1-\frac{z}{w_1}\right)}\right\}$$
$$\times\left\{[2]^{-1}:\phi(w_1)\phi(w_2):\left(\frac{w_1-q^{-2}w_2}{-q^2z\left(1-\frac{w_2}{q^4w_1}\right)}+\frac{1-\frac{w_1}{q^2w_2}}{1-\frac{z}{w_2}}\right)\right.$$

$$+\frac{(w_1w_2)^{1/2}\left(1-\frac{w_2}{w_1}\right)}{-q^2z\left(1-\frac{q^2w_2}{w_1}\right)\left(1-\frac{w_2}{q^4z}\right)}-\frac{\left(\frac{w_1}{w_2}\right)^{1/2}\left(1-\frac{w_1}{w_2}\right)}{\left(1-\frac{q^2w_1}{w_2}\right)\left(1-\frac{z}{w_2}\right)}\Bigg\}. \tag{40}$$

We have to prepare two contours because of the fermionic part and one is for the term including $:\phi(w_1)\phi(w_2):$ and the other is for the rest. The former satisfies $|\frac{w_2}{q^4w_1}|<1, |w_2|>|z|$ and the same condition satisfied by the contour for Ψ_1 with substitution $w=w_1$. The latter satisfies $|q^2w_2|<|w_1|<|q^{-2}w_2|$ and the same conditions as Ψ_1 with $w=w_1,w_2$

$$F_{II,<}(z)=\exp\left(-\sum_{m>0}\frac{a_{-m}}{[2m]}(qz)^m\right) \tag{41}$$

$$F_{II,>}(z)=\exp\left(\sum_{m>0}\frac{a_m}{[2m]}(q^3z)^{-m}\right). \tag{42}$$

Under the normalization

$$\langle 2\Lambda_0|\tilde{\Psi}_0(z)|2\Lambda_1\rangle=1 \qquad \langle 2\Lambda_1|\tilde{\Psi}_2(z)|2\Lambda_0\rangle=1$$
$$\langle \Lambda_0+\Lambda_1|\tilde{\Psi}_1(z)|\Lambda_0+\Lambda_1\rangle=1$$

$\tilde{\Psi}(z)$'s are given by

$$\tilde{\Psi}^{2,2\Lambda_0}_{2\Lambda_1}(z)=\Psi(z) \tag{43}$$

$$\tilde{\Psi}^{2,\Lambda_0+\Lambda_1}_{\Lambda_0+\Lambda_1}(z)=-(-q^2z)^{-1/2}\Psi(z) \tag{44}$$

$$\tilde{\Psi}^{2,2\Lambda_1}_{2\Lambda_0}(z)=(-q^4z)^{-1}\Psi(z). \tag{45}$$

4. Derivation

Taking $\Phi^{\Lambda_0+\Lambda_1,1}_{2\Lambda_i}(z)$ as an example, we discuss the derivation of the results in the previous section. Other cases can be treated in almost the same way.

4.1. General structure of $\Phi_0(z)$ and $\Phi_1(z)$

Calculating

$$\Delta(x)\Phi(z)=\Phi(z)x$$

for $x=$ Chevalley generators of U and a_n, we get

$$0=[\Phi_1(z),x_0^+]$$
$$K\Phi_1(z)=[\Phi_0(z),x_0^+]$$
$$0=x_0^-\Phi_0(z)-q\Phi_0(z)x_0^-$$
$$\Phi_0(z)=\Phi_1(z)x_0^- -qx_0^-\Phi_1(z) \tag{46}$$
$$0=\Phi_0(z)x_1^- -qx_1^-\Phi_0(z)$$
$$q^3z\Phi_0(z)=\Phi_1(z)x_1^- -q^{-1}x_1^-\Phi_1(z) \tag{47}$$
$$(qzK)^{-1}\Phi_1(z)=[\Phi_0(z),x_{-1}^+]$$
$$0=[\Phi_1(z),x_{-1}^+]$$
$$K\Phi_1(z)K^{-1}=q\Phi_1(z) \tag{48}$$
$$K\Phi_0(z)K^{-1}=q^{-1}\Phi_0(z)$$

$$[a_m, \Phi_1(z)] = (q^5 z)^m \frac{[m]}{m} \Phi_1(z) \tag{49}$$

$$[a_{-m}, \Phi_1(z)] = (q^3 z)^{-m} \frac{[m]}{m} \Phi_1(z). \tag{50}$$

From (48)–(50), we can speculate the form of $\Phi_1(z)$ as

$$\Phi_1(z) = B_{I,<}(z) B_{I,>}(z) \Omega^R_{NS}(z) \mathrm{e}^{\alpha/2} y^{\partial}.$$

To determine y and the fermionic part $\Omega^R_{NS}(z)$, we impose the following conditions on $\Phi_1(z)$

$$\Phi_1(z) x_0^- - q x_0^- \Phi_1(z) = (q^3 z)^{-1} (\Phi_1(z) x_1^- - q^{-1} x_1^- \Phi_1(z))$$
$$0 = [\Phi_1(z), x^+(w)]$$

which can be easily seen from (46), (47) and the proposition of section 4.4 of [12]. Then we have (10), (14)

$$\Phi_1(z) = B_{I,<}(z) B_{I,>}(z) \Omega^R_{NS}(z) \mathrm{e}^{\alpha/2} (-q^4 z)^{\partial/4}$$

$$\phi^R(w) \Omega^R_{NS}(z) = \left(\frac{-q^4 z}{w}\right)^{1/2} \frac{\left(\frac{w}{q^3 z}; q^4\right)_\infty \left(\frac{q^7 z}{w}; q^4\right)_\infty}{\left(\frac{w}{qz}; q^4\right)_\infty \left(\frac{q^5 z}{w}; q^4\right)_\infty} \Omega^R_{NS}(z) \phi^{NS}(w).$$

$\Phi_1(z)$ can be calculated through (46)

$$\begin{aligned}\Phi_0(z) &= \oint \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{1}{w} \{\Phi_1(z) x^-(w) - q x^-(w) \Phi_1(z)\} \\ &= \oint \frac{\mathrm{d}w}{2\pi \mathrm{i}} B_{I,<}(z) E^-_<(w) B_{I,>}(z) E^-_>(w) \Omega^R_{NS}(z) \phi^{NS}(w) \mathrm{e}^{-\alpha/2} (-q^4 z)^{\partial/4} \\ &\quad \times w^{-\partial/2} (-q^4 z w^3)^{-\frac{1}{2}} \frac{\left(\frac{w}{q^3 z}; q^4\right)_\infty}{\left(\frac{w}{qz}; q^4\right)_\infty} \left\{ \frac{w}{1 - q^{-3} w/z} + \frac{q^5 z}{1 - q^5 z/w} \right\}.\end{aligned}$$

To determine the contour of integration we have to find the poles of $\Omega^R_{NS}(z) \phi^{NS}(w)$ and this can be seen from

$$\langle R | \Omega^R_{NS}(z) \phi^{NS}(w) | NS \rangle = \frac{\left(\frac{w}{qz}; q^4\right)_\infty}{\left(\frac{w}{q^3 z}; q^4\right)_\infty}$$

$$\langle NS | \Omega^{NS}_R(z) \phi^R(w) | R \rangle = \left(\frac{w}{-q^4 z}\right)^{1/2} \frac{\left(\frac{w}{qz}; q^4\right)_\infty}{\left(\frac{w}{q^3 z}; q^4\right)_\infty}.$$

Hence as a composite $\Omega^R_{NS}(z) \phi^{NS}(w) \frac{\left(\frac{w}{q^3 z}; q^4\right)_\infty}{\left(\frac{w}{qz}; q^4\right)_\infty}$ in the integrand has no poles and the contour is the one encloses $w = 0, q^5 z$.

4.2. *Fermion emission vertex operator*

In [6], equation (15) appears in the study of the Ising model and its free field realization is given without any details. Thus we give the exposition of its derivation†. The main point of

† We are indebted to M Jimbo for explaining the details of [6].

derivating free field realization of the fermion emission vertex operator $\Omega^{R}_{NS}(z)$ (15), (16) is to expand $\Omega^{R}_{NS}(z)$ as

$$\Omega^{R}_{NS}(z) = \sum_{K,L} a_{K,L}\phi^{R}_{k_1}\phi^{R}_{k_2}\ldots|R\rangle\langle NS|\phi^{NS}_{l_1}\phi^{NS}_{l_2}\ldots$$
$$K = \{k_i\} \qquad L = \{l_i\}$$

and to calculate the coefficients $a_{K,L}$. After normalizing ϕ_n suitably to φ_n (19), (20), we see '$a_{K,L}$/(normalization factor)' are identified with Pfaffians of $X_{k,l}$. With the aid of a relation satisfied by Pfaffian

$$\omega^{\wedge n} = n!\mathrm{Pf}(b_{ij})x_1 \wedge x_2 \ldots \wedge x_{2n}$$

where $x_k (1 \leqslant k \leqslant 2n)$ is a Grassmann variable and

$$\omega = \sum_{1\leqslant i<j\leqslant 2n} b_{ij}x_i \wedge x_j$$

we get (15), (16).

Wick's theorem can be generalized to the present situation and we only need to calculate one- and two-point correlation functions for $a_{K,L}$. To calculate these, we rewrite (14) and introduce auxiliary operators

$$\tilde{\phi}^{NS}(w)\Omega^{NS}_{R}(q^{-4}) = \Omega^{NS}_{R}(q^{-4})\tilde{\phi}^{R}(w) \tag{51}$$

$$\tilde{\phi}^{NS}(w) = (-1)^{-1/2}w^{1/2}\frac{(qw^{-1};q^4)_\infty}{(q^3w^{-1};q^4)_\infty}\phi^{NS}(w) \tag{52}$$

$$\tilde{\phi}^{R}(w) = \frac{(qw;q^4)_\infty}{(q^3w;q^4)_\infty}\phi^{R}(w) = f_+(w)\phi^{R}(w). \tag{53}$$

We set $\Omega(z = q^4)$ for simplicity. They are defined to satisfy

$$\langle NS|\tilde{\phi}^{NS}_n = 0 (n<0) \qquad \tilde{\phi}^{R}_n|R\rangle = 0(n>0) \qquad \tilde{\phi}^{R}_0|R\rangle = |R\rangle$$

and this enables us to see that

$$\langle NS|\Omega^{NS}_{R}(q^{-4})\tilde{\phi}^{R}(z)\tilde{\phi}^{R}(w)|NS\rangle = \langle NS|\tilde{\phi}^{NS}(z)\Omega^{NS}_{R}(q^{-4})\tilde{\phi}^{R}(w)|NS\rangle$$

contains only negative (positive) powers of $z(w)$. On the other hand the expectation value of

$$\{\tilde{\phi}^{R}(z), \tilde{\phi}^{R}(w)\} = f_+(z)f_+(w)\left(\delta\left(\frac{q^2w}{z}\right) + \delta\left(\frac{w}{q^2z}\right)\right)$$
$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n$$

with respect to $\langle NS|\Omega^{NS}_{R}(q^{-4})$ and $|R\rangle$ is

$$\langle NS|\Omega^{NS}_{R}(q^{-4})\tilde{\phi}^{R}(z)\tilde{\phi}^{R}(w)|NS\rangle + \langle NS|\Omega^{NS}_{R}(q^{-4})\tilde{\phi}^{R}(w)\tilde{\phi}^{R}(z)|NS\rangle$$
$$= f_+(z)f_+(w)\left(\delta\left(\frac{q^2w}{z}\right) + \delta\left(\frac{w}{q^2z}\right)\right)$$

where we normalize $\langle NS|\Omega^{NS}_{R}(q^{-4})|R\rangle = 1$. We then get

$$\langle NS|\Omega^{NS}_{R}(q^{-4})\tilde{\phi}^{R}(z)\tilde{\phi}^{R}(w)|R\rangle = \frac{1-qw}{1-q^2w/z} + \frac{1-q^{-1}w}{1-q^{-2}w/z} - 1.$$

Expanding the last line of the following equation as in appendix C

$$\langle NS|\Omega_R^{NS}(q^{-4})\phi^R(z)\phi^R(w)|R\rangle = \sum_{n,m\in\mathbb{Z}} \langle NS|\Omega_R^{NS}(q^{-4})\phi_n^R\phi_m^R|R\rangle z^{-n}w^{-m}$$

$$= \frac{1}{f_+(z)f_+(w)}\left\{\frac{1-qw}{1-q^2w/z} + \frac{1-q^{-1}w}{1-q^{-2}w/z} - 1\right\}$$

we have

$$\langle NS|\Omega_R^{NS}(q^{-4})\phi_{-n}^R\phi_{-m}^R|R\rangle = X_{m,n}\gamma_n\gamma_m q^{n+m} \qquad (n, m \geqslant 0). \tag{54}$$

Similar calculation yields

$$\langle NS|\phi_{k+1/2}^{NS}\Omega_R^{NS}(q^{-4})\phi_{-n}^R|R\rangle = -(-1)^{1/2}X_{-k-1/2,n}\gamma_n\gamma_k q^{n+k} \qquad (n, k \geqslant 0) \tag{55}$$

$$\langle NS|\phi_{k+1/2}^{NS}\phi_{l+1/2}^{NS}\Omega_R^{NS}(q^{-4})|R\rangle = -X_{l+1/2,k+1/2}\gamma_l\gamma_k q^{l+k} \qquad (k, l \geqslant 0). \tag{56}$$

z-dependence of $\Omega_{NS}^R(z)$ is recovered with the equation

$$\begin{aligned} &\zeta^{d^R}\Omega_{NS}^R(z)\zeta^{-d^{NS}} = \Omega_{NS}^R(\zeta^{-1}z) \\ &\zeta^{-d^i}\phi^i(z)\zeta^{d^i} = \phi^i(\zeta z) \\ &\langle i|d^i = d^i|i\rangle = 0 \end{aligned} \tag{57}$$

where d^i's are the fermionic part of d of (3)

$$d^i = -\sum_{k>0} kN_k^{\phi^i} \qquad (i = NS \text{ or } R)$$

and satisfy

$$[d^i, \phi_n^i] = n\phi_n.$$

To derive (57), we multiply (14) by ζ^{d^R}, $\zeta^{-d^{NS}}$ from left and right respectively.

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Appendix A. Boson

The following are useful formulae for normal ordering bosons. We set $(z)_\infty = (z; q^4)_\infty$ for brevity.

$$B_{I,>}(z)E_<^-(w) = \frac{(qw/z)_\infty}{(q^{-1}w/z)_\infty}E_<^-(w)B_{I,>}(z)$$

$$E_>^-(w)B_{I,<}(z) = \frac{(q^9z/w)_\infty}{(q^7z/w)_\infty}B_{I,<}(z)E_>^-(w)$$

$$B_{I,>}(z)E_<^+(w) = \frac{(q^{-3}w/z)_\infty}{(q^{-1}w/z)_\infty}E_<^+(w)B_{I,>}(z)$$

$$E_>^+(w)B_{I,<}(z) = \frac{(q^5z/w)_\infty}{(q^7z/w)_\infty}B_{I,<}(z)E_>^+(w)$$

$$B_{II,>}(z)E_<^+(w) = \frac{(q^{-1}w/z)_\infty}{(q^{-3}w/z)_\infty}E_<^+(w)B_{II,>}(z)$$

$$E^+_>(w)B_{II,<}(z) = \frac{(q^3z/w)_\infty}{(qz/w)_\infty} B_{II,<}(z)E^+_>(w)$$
$$B_{II,>}(z)E^-_<(w) = \frac{(q^{-1}w/z)_\infty}{(qw/z)_\infty} E^-_<(w)B_{II,>}(z)$$
$$E^-_>(w)B_{II,<}(z) = \frac{(q^3z/w)_\infty}{(q^5z/w)_\infty} B_{II,<}(z)E^-_>(w)$$
$$F_{II,>}(z)E^-_<(w) = \left(1 - \frac{w}{q^2z}\right) E^-_<(w)F_{II,>}(z)$$
$$E^-_>(w)F_{II,<}(z) = \left(1 - \frac{q^2z}{w}\right) F_{II,<}(z)E^-_>(w)$$
$$F_{II,>}(z)E^+_<(w) = \frac{1}{1 - q^{-4}w/z} E^+_<(w)F_{II,>}(z)$$
$$E^+_>(w)F_{II,<}(z) = \frac{1}{1 - z/w} F_{II,<}(z)E^-_>(w)$$
$$E^-_>(w_1)E^+_<(w_2) = \frac{1}{1 - w_2/w_1} E^+_<(w_2)E^-_>(w_1)$$
$$E^+_>(w_2)E^-_<(w_1) = \frac{1}{1 - w_1/w_2} E^-_<(w_1)E^+_>(w_2).$$

Appendix B. Fermion

For $\Omega^{NS}_R(z)$, we show the equations corresponding to the ones from (51) to (56)

$$\tilde{\phi}^{R'}(w)\Omega^R_{NS}(q^{-4}) = \Omega^R_{NS}(q^{-4})\tilde{\phi}^{NS'}(w) \tag{58}$$
$$\tilde{\phi}^{R'}(w) = \frac{(q/w;q^4)_\infty}{(q^3/w;q^4)_\infty}\phi^R(w) \tag{59}$$
$$\tilde{\phi}^{NS'}(w) = (-1)^{1/2}w^{-1/2}\frac{(qw;q^4)_\infty}{(q^3w;q^4)_\infty}\phi^{NS}(w) \tag{60}$$
$$\langle R|\tilde{\phi}^{R'}_n = 0 (n<0) \qquad \langle R|\tilde{\phi}^{R'}_0 = \langle R| \qquad \tilde{\phi}^{NS'}_n|NS\rangle = 0 \qquad (n>0)$$
$$\langle R|\tilde{\phi}^{R'}(z)\tilde{\phi}^{R'}(w)\Omega^R_{NS}(q^{-4})|NS\rangle = \frac{1-q/z}{1-q^2w/z} + \frac{1-q^{-1}/z}{1-q^{-2}w/z} - 1$$
$$\langle R|\phi^R_n\phi^R_m\Omega^R_{NS}(q^{-4})|NS\rangle = X_{n,m}\gamma_n\gamma_m q^{n+m} (n,m \geqslant 0)$$
$$\langle R|\phi^R_n\Omega^R_{NS}(q^{-4})\phi^{NS}_{-k-1/2}|NS\rangle = (-1)^{1/2}X_{-k-1/2,n}\gamma_n\gamma_k q^{n+k} \qquad (n,k\geqslant 0)$$
$$\langle R|\Omega^R_{NS}(q^{-4})\phi^{NS}_{-k-1/2}\phi^{NS}_{-l-1/2}|NS\rangle = X_{l+1/2,k+1/2}\gamma_l\gamma_k q^{l+k} \qquad (k,l\geqslant 0).$$

Appendix C. Calculation of equation (54)

We show details of calculation of (54). From (21)

$$\langle NS|\Omega^{NS}_R(q^{-4})\phi^R(z)\phi^R(w)|R\rangle = \frac{1}{f_+(z)f_+(w)}\left\{\frac{1-qw}{1-q^2w/z} + \frac{1-q^{-1}w}{1-q^{-2}w/z} - 1\right\} = \sum_{k\geqslant 0, l\geqslant 0}\gamma_k(qz)^k\gamma_l(qw)^l$$

$$\times \left\{ \sum_{a \geqslant 0} \left((1 - qw) \left(\frac{q^2 w}{z} \right)^a + (1 - w/q) \left(\frac{w}{q^2 z} \right)^a \right) - 1 \right\}$$
$$= \sum_{0 \leqslant a \leqslant m} \gamma_{n+a} \gamma_{m-a} \eta_a q^{n+m} z^n w^m - \sum_{0 \leqslant a \leqslant m-1} \gamma_{n+a} \gamma_{m-a-1} (q^{2a} + q^{-2(a+1)})$$
$$\times q^{n+m} z^n w^m - \gamma_n \gamma_m z^n w^m .$$

Hence the equation to be proved is

$$X_{n,m} \gamma_n \gamma_m = \sum_{0 \leqslant a \leqslant m} \gamma_n \gamma_m \eta_a - \sum_{0 \leqslant a \leqslant m-1} \gamma_{n+a} \gamma_{m-a-1} (q^{2a} + q^{-2(a+1)}) - \gamma_n \gamma_m z^n w^m$$

which is equivalent to

$$X_{n,m} = 1 + (1 - t^{-1})(1 + t^{2n}) \sum_{1 \leqslant a \leqslant m} \frac{(t^{1+2n}; t^2)_{a-1}}{(t^{2+2n}; t^2)_{a-1}} \frac{(t^{2m-2a+2}; t^2)_a}{(t^{2m-2a+1}; t^2)_a} \frac{t^a}{1 - t^{2(n+a)}} \qquad (61)$$

where we set $t = q^2$. It can be proved by induction with respect to k that the summation over $a = m, m-1, \ldots, m-k$ yields

$$t^{m-k} \frac{(t^{1+2n}; t^2)_{m-k-1}}{(t^{2+2n}; t^2)_{m-k-1}} \frac{(t^{2k+2}; t^2)_{m-k}}{(t^{2k+1}; t^2)_{m-k}} \frac{\sum_{j=0}^{k} t^{2j}}{1 - t^{2(n+k)}} .$$

Setting $k = m - 1$ we can see that the right-hand side of (61) is equal to $\frac{t^{2m} - t^{2n}}{1 - t^{2(n+m)}}$.

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